

# Gravitational collapse in non-minimally coupled gravity: finite density singularities and the breaking of the no-hair theorem

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In this work we study the dynamics of gravitational collapse of a homogeneous dust sphere in a model exhibiting a linear non-minimal coupling between matter and curvature. The evolution of the scale factor and the matter density is obtained for different choices of Lagrangean density of matter, highlighting the direct physical relevance of the latter in this theory. Following a discussion of the junction conditions and boundary terms in the action functional, the matching with the outer metric and event horizon are analyzed.

We find that a distinct phenomenology arises when compared with standard results for the Oppenheimer-Snyder collapse, namely the possibility of finite density black holes and the breaking of the no-hair theorem, due to a dependence of the end state of a black hole on the initial radius of the spherical body.

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## I. INTRODUCTION

Two of the major challenges faced by contemporary cosmology are the nature of dark energy [1, 2] and dark matter [3] (or perhaps a unification of the two [4], which account for  $\sim 96\%$  of the Universe, and to gather some insight into a more encompassing theory of gravity. A rather straightforward approach for the latter problem resorts to the substitution of the linear scalar curvature term in the Einstein-Hilbert action with a function of the scalar curvature,  $f(R)$  [5] or other scalar invariants of the theory: extensions relying on a functional dependence of the action on the Gauss-Bonnet invariant  $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$  are the most well studied models, given their invoked origin in a low-energy effective description of String Theory and strong implications in braneworld scenarios [6].

The more tractable class of  $f(R)$  theories have had considerable success in replicating the early period of rapid expansion of the universe, as shown by the Starobinsky inflationary model  $f(R) = R + \alpha R^2$  [7]. At late times, the accelerated expansion of the Universe has also been addressed suitably [8]. Solar system tests, mostly arising from the parameterized post-Newtonian (PPN) metric coefficients derived from this extension of General Relativity (GR), have also been discussed [9]. A clear phenomenological consequence of  $f(R)$  theories is the addition of an increasing, repulsive contribution to the Newtonian potential, for power law terms  $f(R) = f_0 R^n$  [10]. Aside from the more usual metric affine connection (that is, where the affine connection is taken *a priori* as de-

pending on the metric), the so-called Palatini approach [11] (where both the metric and the affine connection are taken as independent variables) has also been considered.

Further expanding on this elegant generalization of GR, another interesting possibility has arisen: that the coupling between matter and geometry in the Einstein-Hilbert action is non-minimal — *i.e.* not enforced solely by the invariant  $\sqrt{-g}d^4x$  and the use of the metric to raise and lower indexes and the associated covariant derivative. A non-minimal coupling would imply that geometric quantities (such as the scalar invariants) could explicitly appear in the action [12] (see also Ref. [13] for an early proposal in the context of Riemann-Cartan geometry).

Indeed, one is motivated to do so by the presence of a non-minimal coupling stemming from one-loop vacuum-polarization effects in the formulation of Quantum Electrodynamics in a curved spacetime [14], as well as in the context of scalar-tensor theories, when considering matter scalar fields [15]. Furthermore, it has been shown that a non-minimally coupled model cannot follow the usual procedure establishing the equivalence between  $f(R)$  and a single scalar-tensor theory [16]; indeed, while a model with two scalar fields may describe the same dynamics as Eq. (1), it still requires one of these fields to appear non-minimally coupled with the matter Lagrangean density [16–20].

The presence of non-minimal coupling leads to several phenomenological consequences: in particular, it implies that regions with extreme curvature could lead to considerable deviations from the dynamics predicted by Einstein's theory [12]. A wide range of results has unfolded, including the impact on solar observables [21], axisymmetric astrophysical solutions [22], the possibility to account for galactic [23] and cluster dark matter [24], a mechanism for mimicking a Cosmological Constant at astrophysical scales [17], post-inflationary reheating

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[25] or the current accelerated expansion of the universe [26] (also including the so-called “extended quintessence” [27]). Finally, a thorough discussion of the relevance of the choice of Lagrangean density for a perfect fluid and its direct impact on physical observables was discussed in Ref. [28]. This choice will be of the utmost relevance in the current work.

In this study, one addresses how a non-minimal coupling modifies gravitational collapse; similar studies have been performed in the case of standard  $f(R)$  theories (*i.e.* with  $f_1(R) = f(R) \neq R$  and  $f_2(R) = 1$ ) [29–31]. Several simplifications are made, namely that the collapsing body is purely spherical body composed of a homogeneous distribution of dust — similarly to the well-known Oppenheimer-Snyder (OS) collapse [32] thoroughly studied in GR. Also, the simplest, linear non-minimal coupling  $f_2(R) \sim R$  is considered, and a trivial curvature term  $f_1(R) = R$  is taken so to highlight the effect of the former on the collapse.

## II. THE MODEL

One considers a model that exhibits a non-minimal coupling between geometry and matter, as expressed in the action functional [12],

$$S = \int [\kappa f_1(R) + f_2(R)\mathcal{L}] \sqrt{-g} d^4x . \quad (1)$$

Variation with respect to the action yields the modified Einstein field equations,

$$\begin{aligned} \left( F_1 + \frac{1}{\kappa} F_2 \mathcal{L} \right) G_{\mu\nu} &= \frac{1}{2\kappa} f_2 T_{\mu\nu} + \\ \Delta_{\mu\nu} \left( F_1 + \frac{1}{\kappa} F_2 \mathcal{L} \right) &+ \frac{1}{2} g_{\mu\nu} \left( f_1 - F_1 R - \frac{1}{\kappa} F_2 R \mathcal{L} \right) , \end{aligned} \quad (2)$$

with  $\kappa = c^4/16\pi G$  and  $\Delta_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - g_{\mu\nu} \square$ . The energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} . \quad (3)$$

As expected, GR is recovered from Eq. (3) by setting  $f_1(R) = R$  and  $f_2(R) = 1$ .

The trace of Eq. (3) reads

$$\left( F_1 + \frac{1}{\kappa} F_2 \mathcal{L} \right) R = \frac{1}{2\kappa} f_2 T - 3\square \left( F_1 + \frac{1}{\kappa} F_2 \mathcal{L} \right) + 2f_1 , \quad (4)$$

where  $T$  is the trace of the energy-momentum tensor.

Resorting to the Bianchi identities, one concludes that the energy-momentum tensor of matter may not be covariantly conserved, as

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2} (g^{\mu\nu} \mathcal{L} - T^{\mu\nu}) \nabla_\mu R . \quad (5)$$

Again, in the absence of a non-minimal coupling,  $f_2(R) = 1$  and the covariant conservation of the energy-momentum tensor is recovered. This feature implies that the motion of the matter distribution described by a Lagrangian density  $\mathcal{L}$  does not follow a geodesic curve. Clearly, a violation of the Equivalence Principle may emerge if the *r.h.s.* of the last equation varies significantly for different matter distributions, which suggests a method of testing the model and imposing constraints on the associated couplings. This feature is a fundamental characteristic of a non-minimally coupled model, as shown in Ref. [18].

## III. GRAVITATIONAL COLLAPSE OF A HOMOGENEOUS FLUID

### A. Linear non-minimal coupling

In Ref. [28], it was argued that the correct Lagrangean density  $\mathcal{L}$  for a perfect fluid is  $\mathcal{L} = -\rho$ , as the non-minimal coupling disables the usual on-shell equivalence with other forms (such as  $\mathcal{L} = p$ ). Notwithstanding, in this study one aims at addressing both forms, so that the impact of choosing a particular Lagrangean density on gravitational collapse may be gauged directly.

Since one considers a homogeneous dust distribution, *i.e.*  $\rho = \text{const.}$ ,  $p = 0$ , this is attained by writing  $\mathcal{L} = -\alpha\rho$ , with  $\alpha = 0 \rightarrow \mathcal{L} = p$  and  $\alpha = 1 \rightarrow \mathcal{L} = -\rho$ . A dust distribution implies that there is no supporting pressure to prevent collapse, with no shell crossing during the later (and no exchange of momentum).

Inserting the above expression, together with a trivial curvature term  $f_1(R) = R$  and a linear non-minimal coupling

$$f_2(R) = 1 + \frac{\epsilon}{\kappa} R , \quad (6)$$

and the energy-momentum tensor for a dust distribution,

$$T_{\mu\nu} = \rho u_\mu u_\nu , \quad (7)$$

where the velocity  $u^\mu$  obeys  $u^\mu u_\mu = -1$  and  $u^\mu \nabla_\nu u_\mu = 0$ , one finds that Eq. (3) reads

$$\begin{aligned} \left( 1 - \frac{\epsilon\alpha}{\kappa^2} \rho \right) G_{\mu\nu} &= \\ \frac{1}{2\kappa} \left( 1 + \frac{\epsilon}{\kappa} R \right) \rho u_\mu u_\nu &+ \frac{1}{2} \frac{\epsilon\alpha}{\kappa^2} g_{\mu\nu} R - \frac{\epsilon\alpha}{\kappa^2} \Delta_{\mu\nu} \rho , \end{aligned} \quad (8)$$

Its trace yields

$$R = \frac{\kappa\rho - 3\epsilon\alpha\Box\rho}{2\kappa^2 + \epsilon(2\alpha - 1)\rho} . \quad (9)$$

The non-conservation of the energy-momentum tensor Eq. (5) becomes

$$\nabla_\mu T^{\mu\nu} = -\frac{\epsilon}{\kappa + \epsilon R} (\alpha g^{\mu\nu} + u^\mu u^\nu) \rho \nabla_\mu R . \quad (10)$$

#### IV. ADAPTABILITY OF THE FRW METRIC

In GR, the study of OS collapse is tantamount to the determination of the evolution of the scale factor  $a(t)$  appearing in the Friedmann-Robertson-Walker (FRW) metric, as given by the line element

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] . \quad (11)$$

where  $k$  is the (negative) spatial curvature.

The use of the above metric implies that one has a position-independent scalar curvature  $R$ : naturally, this stems from the identification  $R = \rho(t)$  that arises from the trace of the unperturbed Einstein equations. Naively, one could expect that in the current scenario a homogeneous density also gives rise to a scalar curvature exhibiting only a time dependence, albeit a more convoluted one, as seen in Eq. (9). This, however, fails to acknowledge that a more general metric could give rise to a radial dependence of  $R$ , via the terms involving the (space dependent) metric components appearing in the D'Alembertian operator  $\Box$ .

To assess this, one therefore adopts a more general metric, given by

$$ds^2 = -dt^2 + U(r, t)dr^2 + V(r, t)(d\theta^2 + \sin^2\theta d\phi^2) , \\ U(r, t) = A_1^2(t)h(r) , \quad V(r, t) = A_2^2(t)r^2 , \quad (12)$$

so that  $t$  measures time according to a in-falling observer (see Ref. [29] for a similar treatment in  $f(R)$  theories). One computes the required differential operator,

$$\Box\rho(t) = -\ddot{\rho} - \left( \frac{\dot{A}_1}{A_1} + 2\frac{\dot{A}_2}{A_2} \right) \dot{\rho} , \quad (13)$$

which, by construction, is a function of time only. Thus, from Eq. (9), it becomes clear that the scalar curvature indeed inherits the homogeneity of the matter density distribution,  $R = R(t)$ , and the FRW metric is indeed admissible in the present scenario.

#### V. EVOLUTION OF THE GRAVITATIONAL COLLAPSE

One now derives the equations of motion driving the dynamics of gravitational collapse. With the assumed FRW metric, the scalar curvature given by Eq. (9) becomes

$$R = \frac{\kappa\rho + 3\epsilon\alpha(\ddot{\rho} + 3\frac{\dot{a}}{a}\dot{\rho})}{2\kappa^2 + \epsilon(2\alpha - 1)\rho} \quad (14)$$

Thus, the temporal component of Eq. (10) becomes

$$0 = \dot{\rho} + \left[ 3\frac{\dot{a}}{a} + \frac{\epsilon}{\kappa + \epsilon R}(1 - \alpha)\dot{R} \right] \rho \rightarrow \quad (15) \\ \rho(t) = \rho_0 \left( \frac{\kappa + \epsilon R}{\kappa + \epsilon R_0} \right)^{\alpha-1} \left( \frac{a_0}{a} \right)^3 ,$$

having used  $u^\mu = (1, \vec{0})$  in the adopted comoving coordinates, and where  $_0$  denotes initial values (without loss of generality, one henceforth sets  $a_0 = a(t_0) = 1$ ). Inspection shows that the radial component of Eq. (10) is trivially null.

Substituting Eq. (14) into the above and considering that  $\alpha = 0, 1$  is a “binary” variable yields, after some algebra, the closed expression

$$\rho(t) = \frac{\rho_0 a^{-3}}{1 + \epsilon(1 - \alpha)\frac{\rho_0}{2\kappa^2}(a^{-3} - 1)} . \quad (16)$$

Even without the knowledge of the evolution of the scale factor  $a(t)$ , this result is revealing of the effect of the adopted non-minimal coupling: if  $\mathcal{L} = -\rho \rightarrow \alpha = 1$ , it leads to an unchanged evolution for the increasing density,  $\rho \sim a^{-3}$ .

If, however,  $\mathcal{L} = p \rightarrow \alpha = 0$ , one finds that as the spherical body collapses and  $a(t) \rightarrow 0$ , it is not infinitely compressed,  $\rho \rightarrow \infty$ , but rather it tends towards a final state of finite density  $\rho_f = 2\kappa^2/\epsilon \sim M_P^4/\epsilon$ , where  $M_P$  is the Planck mass. Furthermore, if  $\epsilon \gg 1$ , this density is low enough that no (yet unknown) quantum effects should arise and one is left with a classical object of vanishing size but finite density!

Such counter-intuitive result is of course related with the non-conservation of energy, the most striking feature of non-minimally coupled models. Also, notice that the choice of Lagrangean density indeed has a crucial role in the physical outcome of the model, as discussed in the previous section.

Replacing Eq. (14) into the modified field Eqs. (9) and following some tedious computations, one obtains the two dynamical equations of motion relating the density with the scale factor and the spatial curvature,

$$0 = (\kappa^2 + 2\epsilon\alpha\rho)(k + 2a\ddot{a}) + (\kappa^2 - 4\epsilon\alpha\rho)(\dot{a})^2 ,$$

$$\frac{1}{6}\kappa a^2 \rho = (\kappa^2 - \epsilon \rho) k + [\kappa^2 + (3\alpha - 1)\epsilon \rho] (\dot{a})^2 + \epsilon(\alpha - 1)a\ddot{a}\rho . \quad (17)$$

Notice that  $\alpha = 0$  leads to the presence of the second derivative of the scale factor in the second equation. Evaluating the first one at  $t = t_0$ , and assuming a collapse with initial null velocity,  $\dot{a}(0) = 0$ , one gets  $k = -2\ddot{a}_0 > 0$ , regardless of  $\alpha$  (with  $\ddot{a}_0 \equiv \ddot{a}(0)$ ) — unless  $\alpha = 1$  and  $\rho_0 = -\kappa^2/2\epsilon$  (implying a negative coupling strength  $\epsilon$ ), in which case  $k$  remains undefined.

One may use Eq. (17) to eliminate  $\ddot{a}$  and write

$$\frac{1}{3}\kappa a^2 \rho = [2\kappa^2 - (1 + \alpha)\epsilon \rho] k + [2\kappa^2 + (5\alpha - 1)\epsilon \rho] (\dot{a})^2 . \quad (18)$$

which, again assuming  $\dot{a}(0) = 0$ , yields

$$k = \frac{1}{3\kappa} \frac{\rho_0}{2 - (1 + \alpha)\epsilon_0} , \quad (19)$$

defining the dimensionless parameter  $\epsilon_0 = \epsilon \rho_0 / \kappa^2$ . Thus, one finds that the non-minimal coupling induces a shift from the value for the spatial curvature found in GR,  $k_0 = \rho_0 / 6\kappa$ . Moreover, a positive spatial curvature also implies that the initial density and the coupling strength are constrained by

$$\rho_0 < \frac{\kappa^2}{\epsilon} \frac{2}{(1 + \alpha)} . \quad (20)$$

Substituting the expression for  $\rho(t)$  found in Eq. (16) into the first of Eqs. (17) and using the identity  $\alpha(\alpha - 1) = 0$ , one obtains

$$0 = (\kappa^2 + 2\epsilon\alpha\rho_0 a^{-3}) (k + 2a\ddot{a}) + (\kappa^2 - 4\epsilon\alpha\rho_0 a^{-3}) (\dot{a})^2 , \quad (21)$$

leading to a simplified equation of motion for the scale factor

$$\dot{a} = -\sqrt{k(1 - a) \frac{a^2 + (a + 1)\alpha\epsilon_0}{a^3 + 2\alpha\epsilon_0}} , \quad (22)$$

supplemented by the initial condition  $a(0) = 1$ .

One may easily check that when  $a \sim 0$  the scalar curvature behaves as

$$R \simeq (1 + 2\alpha) \frac{3k}{a^2} . \quad (23)$$

so that both choices of Lagrangean density lead to a singularity.

Defining the cycloid parameter  $\eta$  as,

$$d\eta = \frac{\sqrt{k}}{a} \sqrt{\frac{a^3 + a(a + 1)\alpha\epsilon_0}{a^3 + 2\alpha\epsilon_0}} dt , \quad (24)$$

one finds that Eq. (22) becomes the usual evolution equation found in GR,

$$a'(\eta) = -\sqrt{a(1 - a)} , \quad a(0) = 1. \quad (25)$$

The effect of the non-minimal coupling is ascribed to the relation between the cycloid time  $\eta$  and the coordinate time  $t$ : if  $\mathcal{L} = p \rightarrow \alpha = 0$ , this amounts only to a shift of the spatial curvature  $k$ ; if  $\mathcal{L} = -\rho \rightarrow \alpha = 1$ , Eq. (24) leads to a more complex relation between the latter.

As in GR, the solution of Eq. (25) is

$$a(\eta) = \frac{1 + \cos(\eta)}{2} , \quad (26)$$

so that the gravitational collapse ends when  $\eta = \pi$ .

## VI. BOUNDARY MATCHING

Assuming that space outside the collapsing spherical body is empty, Eq. (3) become the trivial vacuum equations  $R_{\mu\nu} = 0$  found in GR, and as such spacetime is described by a Schwarzschild metric. The outer region is endowed with a coordinate system that does not coincide with the one used in the interior region: from spherical symmetry, one sees that only the time and radial coordinates will be different. Labeling these as  $t'$  and  $r'$ , respectively, one writes the outer metric via the usual line element

$$ds^2 = -\left(1 - \frac{2GM}{r'}\right) dt'^2 + \left(1 - \frac{2GM}{r'}\right)^{-1} dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2) . \quad (27)$$

For the full spacetime to be well defined, one must ensure that the FRW metric valid Eq. (11) inside the collapsing body matches smoothly with this outer Schwarzschild metric.

One first requires that the induced metric on the boundary is equal on both sides, as any discontinuities would lead to an ill-defined scalar curvature  $R$  — thus formulating the so-called first junction condition. For this, one first defines the boundary of the collapsing spherical body as a spacelike hypersurface given by the condition  $r = r_* = \text{const.}$  (since  $r$  is a comoving coordinate) or  $r' = R_*(t')$  (*i.e.* an external observer sees the boundary receding towards  $R_* = 0$ ).

The tangent vectors are given by

$$e_a^\alpha = \frac{\partial y^\alpha}{\partial x^a} , \quad (28)$$

where  $y^\alpha$  are four coordinates used in the inner and outer metrics and  $x^a$  ( $a = t, \theta, \phi$ ) are the three coordinates

parameterizing the boundary [33]. The induced metric  $h_{ab}$  on this hypersurface, defined as,

$$h_{ab} = g_{\alpha\beta} e_a^\alpha e_b^\beta, \quad (29)$$

is then given by the line element

$$ds_\Sigma^2 = -dt^2 + a^2 r_*^2 (d\theta^2 + \sin^2 \theta d\phi^2) = - \left[ 1 - \frac{2GM}{R_*} - \left( 1 - \frac{2GM}{R_*} \right)^{-1} \left( \frac{dr'}{dt'} \right)^2 \right] dt'^2 + R_*^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (30)$$

By inspection, one obtains the matching conditions for the inner and outer coordinate systems,

$$R_* = ar_* , \quad (31)$$

$$\dot{t}' = \frac{\sqrt{R_*[R_* - 2GM + R_*(\dot{R}_*)^2]}}{R_* - 2GM}.$$

where the dot still indicates differentiation with respect to  $t$ .

The smooth crossover between the inner and outer description of spacetime also demands that the derivatives of the corresponding metrics are properly matched. A quick and dirty approach is to require that the derivatives with respect to the radial coordinates are equal; however, this procedure is explicitly coordinate-dependent. A more elegant, coordinate invariant approach relies on the computation of the (dis)continuity conditions for the extrinsic curvature tensor  $K_{ab}$  [33].

In GR, this ensuing second junction condition (in the absence of a thin shell, see Ref. [34] for a related discussion) translates into the equality  $K_{ab}^- = K_{ab}^+$ . Erroneously, some assume that this is a universal statement born from geometrical necessity or aesthetic considerations: for example, in Ref. [30], the unwarranted use of this equality in the context of  $f(R)$  theories leads to a time dependent mass  $M(t')$ , thus producing an unphysical outer vacuum with a non-vanishing, time dependent curvature!

Given the above, one concludes that when GR is modified, the second junction condition may in general read  $[K_{ab}] \equiv K_{ab}^+ - K_{ab}^- \neq 0$  [35], with the *r.h.s.* reflecting the altered structure of the equations of motion.

In order to pursue its correct formulation, two equivalent paths are available, as described schematically below. The first works at the level of the equations of motion and resorts to a functional description of the relevant quantities [33], thus writing the metric as

$$g_{\mu\nu} = H^-(l)g_{\mu\nu}^- + H^+(l)g_{\mu\nu}^+, \quad (32)$$

where  $H^\pm$  is the Heaviside step function and  $l$  is an affine parameter describing the crossing of the hypersurface at

$l = 0$ . In the interior of the spherical body  $l < 0$  and  $H^-(l) = 1$ , while outside  $H^-(l) = 0$  (and conversely for  $H^+$ ).

The computation of the Ricci tensor and the scalar curvature involve derivatives of the metric; recalling that  $H'(l) = \delta(l)$ , one obtains

$$R = H^- R^- + H^+ R^+ + \delta A, \quad (33)$$

where  $A = A(g_{\mu\nu}, g_{\mu\nu,\alpha}, n_\beta)$  and  $n_\beta$  is the unit normal to the hypersurface; the latter enables the relation between the induced and the inner and outer metrics at the boundary,

$$g^{\alpha\beta} = n^\alpha n^\beta + h^{ab} e_a^\alpha e_b^\beta. \quad (34)$$

Following the same procedure for the matter content, one writes its energy-momentum tensor as

$$T_{\mu\nu} = H^-(l)T_{\mu\nu}^- + H^+(l)T_{\mu\nu}^+, \quad (35)$$

with an additional  $\delta(l)$  term if a boundary layer is present.

By inserting the above expansions for the relevant quantities into Eq. (3) and demanding that terms in  $\delta(l)$  vanish, one is in principle able to obtain the desired second junction condition. Since many modifications of GR increase the order of the differential operators in the equations of motion, in general this will lead to added terms in  $\delta$  that yield a discontinuity of the extrinsic curvature across the hypersurface. Also, particular care should be taken with the appearance of crossed terms such as  $H^-\delta$ , as these are ill-defined as functionals — an issue that could be surpassed with the substitution of the Heaviside step and Dirac delta functions by suitable, convergent, approximations.

The second, much more elegant procedure works at the level of the action of the theory. It relies on the rederivation of the field equations from an action defined within a closed volume of space time: when one follows the usual procedure and applies the Gauss-Stokes theorem, this confinement leads to finite terms (as one can no longer evoke that these vanish at infinity). In order to obtain the same equations of motion, these undesired quantities must be countermanded by a suitable boundary contribution, defined solely on the hypersurface — the Gibbons-York-Hawking term [36] (see Ref.[35] for a general derivation in the context of  $f(R)$  models). Finally, by varying the action with respect to the induced metric on the later, one straightforwardly obtains the sought for second junction condition.

In this study, one may quickly conclude as to what method is more profitable: if one adopted the first procedure, the presence of the term  $\Delta_{\mu\nu}\mathcal{L}$  in Eq. (9) would lead to the appearance of derivatives of the Dirac delta, which are functionally defined as  $\delta'(l)f(l) = -\delta f'(l)$ .



This would lead to extremely cumbersome calculations, thus favoring the second approach outlined above.

With the above in mind, one varies Eq. (1) with the adopted forms for  $f_1 = R$  and  $f_2 = 1 + \epsilon R/\kappa$  and the constraint  $\delta g_{\mu\nu} = 0$  in the hypersurface  $\partial V$ . Using

$$\delta R_{\alpha\beta} = \nabla_\sigma [g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\sigma - g^{\alpha\sigma} \delta \Gamma_{\alpha\beta}^\beta] , \quad (36)$$

one gets

$$\begin{aligned} \delta S = \int_V \sqrt{-g} d^4 x \times \\ \left[ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) R_{\alpha\beta} - \frac{1}{2} \kappa R g_{\alpha\beta} - \frac{1}{2} \left( 1 + \epsilon \frac{R}{\kappa} \right) T_{\alpha\beta} \right. \\ \left. + \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\sigma [g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\sigma - g^{\alpha\sigma} \delta \Gamma_{\alpha\beta}^\beta] \right] . \end{aligned} \quad (37)$$

The second line in the integral above, dubbed  $\delta S_2$ , may be integrated via the Gauss-Stokes theorem. For this, one first uses the definition of  $\Gamma_{\alpha\beta}^\gamma$ , obtaining

$$\begin{aligned} \nabla_\sigma \left( g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\sigma - g^{\alpha\sigma} \delta \Gamma_{\alpha\beta}^\beta \right) = \\ g_{\alpha\beta} \square (\delta g^{\alpha\beta}) - \nabla_\alpha \nabla_\beta (\delta g^{\alpha\beta}) . \end{aligned} \quad (38)$$

For convenience, the integral symbols are substituted by an abbreviated notation,

$$\begin{aligned} \int_V X \sqrt{-g} d^4 x &\equiv \{X\}_V , \\ \int_{\partial V} X \sqrt{-h} d^3 x &\equiv \{X\}_{\partial V} , \end{aligned} \quad (39)$$

where  $h$  is the determinant of the induced metric  $h_{ab}$ ; with this notation, the Gauss-Stokes theorem reads

$$\{\nabla_\mu X\}_V = \{n_\mu X\}_{\partial V} . \quad (40)$$

Integrating by parts and twice using this theorem, one gets

$$\begin{aligned} \delta S_2 \equiv \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\sigma [g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\sigma - g^{\alpha\sigma} \delta \Gamma_{\alpha\beta}^\beta] \right\}_V = \\ \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) g_{\alpha\beta} \nabla_\sigma \nabla^\sigma (\delta g^{\alpha\beta}) \right\}_V - \\ \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\alpha \nabla_\beta (\delta g^{\alpha\beta}) \right\}_V = \\ \left\{ \nabla_\sigma \left[ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right] \right\}_V - \\ \left\{ \nabla_\alpha \left[ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\beta (\delta g^{\alpha\beta}) \right] \right\}_V + \\ \left\{ \frac{\epsilon}{\kappa} (\nabla_\alpha \mathcal{L}) \nabla_\beta (\delta g^{\alpha\beta}) \right\}_V - \left\{ \frac{\epsilon}{\kappa} (\nabla_\sigma \mathcal{L}) g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right\}_V = \\ \left\{ n_\sigma \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right\}_{\partial V} - \end{aligned} \quad (41)$$

$$\begin{aligned} \left\{ n_\alpha \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\beta (\delta g^{\alpha\beta}) \right\}_{\partial V} + \\ + \left\{ \frac{\epsilon}{\kappa} \nabla_\beta (\nabla_\alpha \mathcal{L} \delta g^{\alpha\beta}) \right\}_V - \left\{ \frac{\epsilon}{\kappa} \nabla^\sigma (\nabla_\sigma \mathcal{L} g_{\alpha\beta} \delta g^{\alpha\beta}) \right\}_V + \\ \left\{ \frac{\epsilon}{\kappa} (\nabla_\sigma \nabla^\sigma \mathcal{L}) g_{\alpha\beta} \delta g^{\alpha\beta} \right\}_V - \left\{ \frac{\epsilon}{\kappa} (\nabla_\alpha \nabla_\beta \mathcal{L}) \delta g^{\alpha\beta} \right\}_V = \\ \left\{ n_\sigma \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right\}_{\partial V} - \\ \left\{ n_\alpha \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\beta (\delta g^{\alpha\beta}) \right\}_{\partial V} + \\ + \left\{ \frac{\epsilon}{\kappa} n_\beta (\nabla_\alpha \mathcal{L}) \delta g^{\alpha\beta} \right\}_{\partial V} - \left\{ \frac{\epsilon}{\kappa} n_\sigma (\nabla^\sigma \mathcal{L}) g_{\alpha\beta} \delta g^{\alpha\beta} \right\}_{\partial V} + \\ \left\{ \frac{\epsilon}{\kappa} (\nabla_\sigma \nabla^\sigma \mathcal{L}) g_{\alpha\beta} \delta g^{\alpha\beta} \right\}_V - \left\{ \frac{\epsilon}{\kappa} (\nabla_\alpha \nabla_\beta \mathcal{L}) \delta g^{\alpha\beta} \right\}_V = \\ \left\{ n_\sigma \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right\}_{\partial V} - \\ \left\{ n_\alpha \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \nabla_\beta (\delta g^{\alpha\beta}) \right\}_{\partial V} + \\ \left\{ \frac{\epsilon}{\kappa} (\nabla_\sigma \nabla^\sigma \mathcal{L}) g_{\alpha\beta} \delta g^{\alpha\beta} \right\}_V - \left\{ \frac{\epsilon}{\kappa} (\nabla_\alpha \nabla_\beta \mathcal{L}) \delta g^{\alpha\beta} \right\}_V . \end{aligned}$$

having used  $\delta g^{\alpha\beta} = 0$  at the boundary.

Thus, the variation of the full action reads

$$\begin{aligned} \delta S = \delta \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) R_{\alpha\beta} - \frac{1}{2} \kappa R g_{\alpha\beta} - \right. \\ \left. \frac{1}{2} \left( 1 + \epsilon \frac{R}{\kappa} \right) T_{\alpha\beta} - \frac{\epsilon}{\kappa} \Delta_{\alpha\beta} \mathcal{L} \right\}_V + \\ \delta \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) [n_\sigma g_{\alpha\beta} \nabla^\sigma (\delta g^{\alpha\beta}) - n_\alpha \nabla_\beta (\delta g^{\alpha\beta})] \right\}_{\partial V} . \end{aligned} \quad (42)$$

The first two lines yield the modified field Eq. (9); the last line must be balanced by an adequate boundary term, given by

$$\delta S_b = \delta \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) [n_\alpha \partial_\beta (\delta g^{\alpha\beta}) - n_\sigma g_{\alpha\beta} \partial^\sigma (\delta g^{\alpha\beta})] \right\}_{\partial V} . \quad (43)$$

since on the boundary  $\delta g^{\alpha\beta} = 0$  and thus  $\nabla_\sigma (\delta g^{\alpha\beta}) = \partial_\sigma (\delta g^{\alpha\beta})$ .

Since  $\delta g^{\alpha\beta}$  does not vary in the hypersurface, its derivative along the tangent vectors vanishes,

$$e_a^\alpha e_b^\beta h^{ab} \partial_\beta (\delta g_{\sigma\alpha}) = 0 , \quad (44)$$

so that, from Eq. (34), one obtains the simplification

$$\begin{aligned} (g^{\alpha\beta} - n^\alpha n^\beta) \partial_\beta (\delta g_{\sigma\alpha}) = 0 \rightarrow \\ \partial_\alpha (\delta g^{\sigma\alpha}) = n_\alpha n^\beta \partial_\beta (\delta g^{\sigma\alpha}) \end{aligned} \quad (45)$$

implying that

$$\begin{aligned} n_\alpha \partial_\beta (\delta g^{\alpha\beta}) - n_\sigma g_{\alpha\beta} \partial^\sigma (\delta g^{\alpha\beta}) = \\ n_\sigma (n_\alpha n_\beta - g_{\alpha\beta}) \partial^\sigma (\delta g^{\alpha\beta}) = \\ -n^\sigma e_a^\alpha e_b^\beta h_{ab} \partial_\sigma (\delta g^{\alpha\beta}) . \end{aligned} \quad (46)$$

As the variation of the trace of the extrinsic curvature  $K = h^{ab}K_{ab}$  with respect to  $g^{\alpha\beta}$  is given by

$$\delta K = \frac{1}{2}n^\sigma e_\alpha^a e_\beta^b h_{ab} \partial_\sigma (\delta g^{\alpha\beta}) \quad , \quad (47)$$

one gets

$$\delta S_b = -2 \left\{ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) \delta K \right\}_{\partial V} \quad . \quad (48)$$

Since  $2\delta\mathcal{L} = (g_{\alpha\beta}\mathcal{L} - T_{\alpha\beta})\delta g^{\alpha\beta} = 0$  on the boundary, one finally obtains the full action including boundary terms (written again in standard notation),

$$S = \int_V \left[ \kappa R + \left( 1 + \epsilon \frac{R}{\kappa} \right) \mathcal{L} \right] \sqrt{-g} d^4x - \quad (49)$$

$$- 2 \int_{\partial V} \left[ \left( \kappa + \frac{\epsilon}{\kappa} \mathcal{L} \right) K + \epsilon \alpha \kappa x(h_{ab}, \mathcal{L}) \right] \sqrt{-h} d^3x \quad .$$

The term  $x(h_{ab}, \mathcal{L})$  (with dimensions of mass) stems from the fact that one can supplement the boundary terms above with additional contributions involving  $h_{ab}$  and  $\mathcal{L}$  only, as these do not show up when variation with respect to  $g_{\alpha\beta}$  is performed (again, since  $\delta\mathcal{L} = 0$  on the boundary). It is factored by  $\epsilon\alpha$  as one expects it to be present only when these quantities are non-vanishing (*i.e.* one has a non-minimal coupling with  $\mathcal{L} \neq p = 0$ ).

The second junction condition may be obtained by variation of the above expression with respect to  $h^{ab}$  on both sides of the boundary; considering that there is no surface energy-momentum tensor  $S_{ab}$  describing a boundary layer, one has

$$S_{ab} = -\frac{2}{\sqrt{-h}} \frac{\delta(\sqrt{-h}\mathcal{L})}{\delta h^{ab}} = 0 \rightarrow \delta\mathcal{L} = \frac{1}{2}\mathcal{L}h_{ab}\delta h^{ab} \quad . \quad (50)$$

so that, after manipulating the tensors, one obtains

$$K_{ab}^+ = \left( 1 - \frac{\epsilon\alpha}{\kappa^2}\rho \right) K_{ab}^- + \frac{\epsilon\alpha}{\kappa^2}\rho K^- h_{ab} + \quad (51)$$

$$+ \epsilon\alpha [X_{ab} + h_{ab}(x - X)] \quad ,$$

defining  $X_{ab} \equiv \delta x / \delta h^{ab}$  and its trace  $X = h^{ab}X_{ab}$ .

Computing the extrinsic curvature tensor for the inner and outer metric,

$$K_t^{-t} = 0 \quad , \quad (52)$$

$$K_\theta^{-\theta} = K_\phi^{-\phi} = \frac{\sqrt{1 - kr^2}}{ar_*} \quad ,$$

$$K_{t'}^{+t'} = \frac{GM + R_*^2 \ddot{R}_*}{R_* \sqrt{R_* [R_* - 2GM + R_* (\dot{R}_*)^2]}} \quad ,$$

$$K_\theta^{+\theta} = K_\phi^{+\phi} = \frac{\sqrt{R_* [R_* - 2GM + R_* (\dot{R}_*)^2]}}{R_*^2} \quad ,$$

one may now use Eq. (52) to fix the mass  $M$ , related to the Schwarzschild radius  $R_s \equiv 2GM$ ; the former is expected to differ from the gravitational mass, defined as  $M_0 = (4\pi/3)\rho R_*^3$ .

## VII. LAGRANGIAN DENSITY CHOICE $\mathcal{L} = p$

The choice of Lagrangean density  $\mathcal{L} = p \rightarrow 0$  merely leads to a shift in the definition of the cycloid parameter  $\eta$  Eq. (24) via the spatial curvature  $k$ , Eq. (19): thus, the gravitational collapse is dynamically equivalent to the case of GR, with the fundamental difference that it does not lead to a singularity of infinite density, as seen from Eq. (16).

Solving for the original time  $t$ , one obtains

$$\frac{dt}{d\eta} = \frac{a}{\sqrt{k}} = \frac{1 + \cos \eta}{2\sqrt{k}} \rightarrow t = \frac{\eta + \sin \eta}{2\sqrt{k}} \quad . \quad (53)$$

Gravitational collapse ends at a time  $\eta_f = \pi$ , which translates into the usual expression

$$t_f = \frac{\pi}{2\sqrt{k}} = \frac{\pi}{2} \sqrt{\frac{6\kappa}{\rho_0} \left( 1 - \frac{\epsilon_0}{2} \right)} \quad . \quad (54)$$

For a positive coupling strength  $\epsilon$ , one finds that the final state of finite density  $\rho_f$  is attained earlier than in OS collapse, since the spatial curvature is larger.

### A. Apparent and event horizon

Following the preceding section, if  $\mathcal{L} = p \rightarrow \alpha = 0$  one expects that the trapped surfaces, apparent and event horizon all occur analogously to OS collapse, with the non-minimal coupling manifesting itself merely through the shifted spatial curvature  $k$ , Eq. (19). Indeed, by introducing the new radial coordinate  $\chi = \arcsin(\sqrt{k}r)$ , one sees that the metric Eq. (11) becomes

$$ds^2 = \frac{a^2}{k} \left[ -d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad , \quad (55)$$

so that radial photons follow null geodesics which are straight lines in the  $(\eta, \chi)$  plane, as in GR.

The calculation of trapped surfaces and the ensuing apparent and event horizon proceeds accordingly: in particular, the apparent horizon crosses the surface of the star when it has collapsed below the Schwarzschild radius, and becomes fixed at this value.

In order to determine the latter, one resorts to the second junction condition, Eq. (52): setting  $\alpha = 0$ , one sees that the continuity relation  $[K_{ab}] = 0$  is recovered. In particular, using Eq. (52), the  $tt$  component together with the identification  $R_* = ar_*$  yields

$$0 = \frac{GM + R_*^2 \ddot{R}_*}{R_* \sqrt{R_* [R_* - 2GM + R_* (\dot{R}_*)^2]}} \rightarrow \quad (56)$$

$$R_s \equiv 2GM = -2a^2 \ddot{a} r_*^3 = a[k + (\dot{a})^2] r_*^3 = k r_*^3 .$$

having used Eqs. (17) and (22) with  $\alpha = 0$ ; the same result of course arises from  $[K_{\theta\theta}] = 0$ . Inserting Eq. (19), one gets

$$M = \frac{4\pi}{3} \frac{\rho_0}{1 - \epsilon_0/2} r_*^3 = \frac{M_0}{1 - \epsilon_0/2} , \quad (57)$$

showing that the mass of the spherical body, as inferred by an outer observer, is increased due to the presence of the non-minimal coupling.

Remarkably, this result shows that a non-minimal coupling can break the no hair theorem: indeed, two stars with the same gravitational mass but different sizes (*i.e.* initial densities  $\rho_0$ ) will produce black holes with unequal event horizons.

### VIII. LAGRANGEAN DENSITY CHOICE $\mathcal{L} = -\rho$

The choice of Lagrangean density  $\mathcal{L} = p \rightarrow 0$  leads to a more convoluted effect of the non-minimal coupling, as the coordinate time  $t$  is related to the cycloid parameter  $\eta$  through

$$t = \int_0^\eta \frac{1}{2\sqrt{k}} \sqrt{\frac{(1 + \cos \eta) [(1 + \cos \eta)^3 + 16\epsilon_0]}{(1 + \cos \eta)^2 + 2(3 + \cos \eta)\epsilon_0}} d\eta . \quad (58)$$

Since the above cannot be solved analytically, one resorts to a numerical integration, yielding the relation  $\eta(t)$  depicted in Fig. 1. Substituting  $\eta(t)$  into  $a(\eta) = (1 + \cos \eta)/2$  leads to the modified evolution of the scale factor for different values of  $\epsilon_0$ , shown in Fig. 2; notice that a large deviation of  $\eta(t)$  with respect to its GR counterpart arises even for small values of  $\epsilon_0$  (where the shift of the spatial curvature Eq. (19) is negligible), due to the additional term in Eq. (58).

The relative decrease of the collapse time  $t_f$  compared to the elapsed period  $t_{fOS} = \pi/2\sqrt{k_0}$  for the OS scenario is depicted in Fig. 3, with the former being given by the equality  $\eta(t_f) = \pi$ : notice that for large  $\epsilon_0$ , one gets

$$t_f \approx \int_0^\pi \sqrt{\frac{2}{k}} \sqrt{\frac{1 + \cos \eta}{3 + \cos \eta}} d\eta = \frac{\pi}{\sqrt{2k}} , \quad (59)$$

leading to the displayed asymptotic behaviour.

#### A. Matching with outer solution

In the present case  $\mathcal{L} = -\rho$ , one unfortunately finds that the inner FRW metric Eq. (11) cannot be suitably

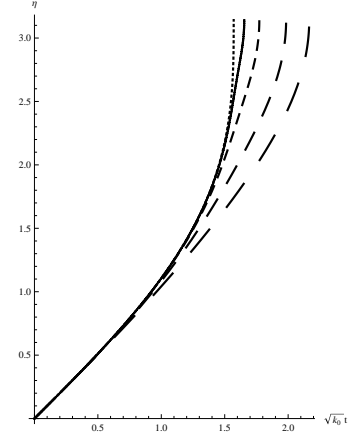


FIG. 1: Evolution of the cycloid time for different values of  $\epsilon_0 = 10^{-3}$  (full),  $\epsilon = 10^{-2}$  (small dash),  $\epsilon = 10^{-1}$  (medium dash) and  $\epsilon = 1$  (large dash); dotted indicates  $\epsilon = 0$ .

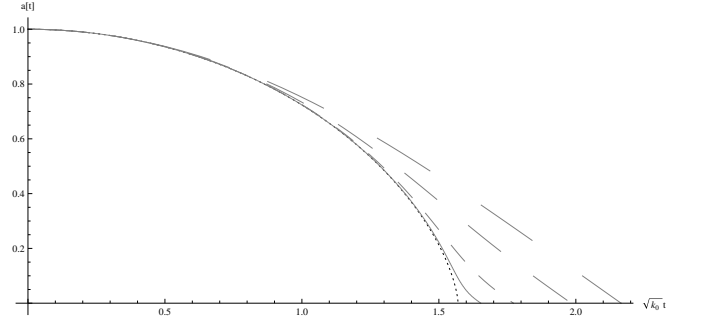


FIG. 2: Evolution of the scale factor for different values of  $\epsilon_0 = 10^{-3}$  (full),  $\epsilon = 10^{-2}$  (small dash),  $\epsilon = 10^{-1}$  (medium dash) and  $\epsilon = 1$  (large dash); dotted indicates  $\epsilon = 0$ .

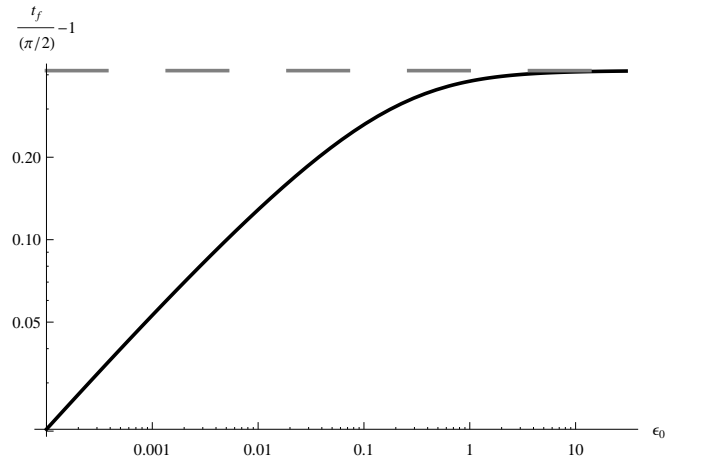


FIG. 3: Relative decrease of the collapse time  $t_e/t_{eOS} - 1$  for different values of  $\epsilon_0$ ; notice the asymptotic behaviour  $t_e \rightarrow \pi/(2\sqrt{k})$ .



embedded with the outer Schwarzschild metric Eq. (27). To ascertain this, one resorts to the junction conditions uncovered before, namely the coordinate matching at the boundary  $R_* = ar_*$  and Eq. (52), here repeated with  $\alpha = 1$ :

$$K_{ab}^+ = \left(1 - \frac{\epsilon}{\kappa^2}\rho\right) K_{ab}^- + \frac{\epsilon}{\kappa^2}\rho K^- h_{ab} + \epsilon [X_{ab} + h_{ab}(x - X)] , \quad (60)$$

By evaluating the above condition, one should be able to read the constant  $M$ . However, this turns out to be unattainable, both for  $x = 0$  as well as for a number of candidates for this extra term to the boundary action Eq. (49), *e.g.*  $x = \rho$  or  $x = h^{ab}\nabla_a\nabla_b\rho$ .

Conversely, one may attempt to solve the above for  $x$ , thus obtaining the additional boundary term that must be considered so that the constant  $M$  is recovered: although this is in principle possible, inspection shows it to be extremely cumbersome, with the foreseeable result producing an extremely convoluted and unfounded expression on  $\rho$ .

### 1. Comparison with OS collapse

In the absence of the non-minimal coupling, there is a much more straightforward way to approach the problem, which indeed produces the same results as the painstaking derivation of the Gibbons-York-Hawking boundary action [36] and the ensuing second junction condition  $[K_{ab}] = 0$ .

In the standard OS collapse, the absence of pressure indicates that the dust particles on the surface of the spherical body are free-falling along radial geodesics of the outer Schwarzschild metric, so that

$$R_* = \frac{R_i}{2}(1 + \cos\eta') , \quad (61)$$

$$\tau = \sqrt{\frac{R_i^3}{8M}}(\eta' + \sin\eta') .$$

where  $\eta'$  is a cycloid parameter related to the proper time  $\tau$  of the infalling observer [37]; the latter is identical with the coordinate time of the FRW metric,  $\tau = t$ .

Recalling the solution Eq. (26) and the definition of the inner cycloid parameter Eq. (24) (here repeated for convenience),

$$a(\eta) = \frac{1 + \cos(\eta)}{2} , \quad (62)$$

$$t = \int \frac{a}{\sqrt{k}} \sqrt{\frac{a^3 + 2\alpha\epsilon_0}{a^3 + a(a+1)\alpha\epsilon_0}} d\eta ,$$

one finds that the relation  $R_* = ar_*$ , stemming from the continuity of the induced metric  $h_{ab}$  across the boundary

(*e.g.* the first junction condition) is only valid for all times in the  $\alpha = 0$  case (see Eq. (53)), with  $R_i = r_*$  (since  $a(0) \equiv 1$ ) and

$$\eta = \eta' , \quad \sqrt{\frac{R_i^3}{8GM}} = \frac{1}{2\sqrt{k}} . \quad (63)$$

The later leads to the result of Eq. (56),  $2GM = kr_*^3$ , valid both for GR (with  $M = M_0$ ) as well as the non-minimally coupled  $\alpha = 0$  case.

Following this approach, one traces the impossibility of recovering an expression for  $M$  when  $\alpha = 1$  to the mismatch between the definitions of the cycloid parameters Eq. (24) and Eq. (61). In its turn, this signals a fault in one of the assumptions of the procedure depicted above — namely that dust particles on the surface of the spherical body free-fall according to radial geodesics of the outer metric.

Indeed, in GR this stems from the condition of vanishing pressure,  $p = 0$ ; in the non-minimally coupled scenario, an effective pressure arises, as the  $rr$  component of the modified field equations does not vanish,  $p_{eff} \equiv 2\kappa g^{rr}G_{rr} \neq 0$ , for  $\alpha = 1$  — and as a result dust particles in the surface experience an additional force that displaces them with respect to outer radial geodesics. It is null for  $\alpha = 0$ , so that the above discussion is valid, as attested by the matching between the inner and outer cycloid parameters.

This is more than a simple mathematical curiosity of the scenario under scrutiny, as it recalls a similar problem in GR: the impossibility of matching the inner and outer spacetimes in the case of a gravitational collapse of a homogeneous sphere  $\rho = \rho(t)$  with non-vanishing pressure  $p \neq 0$ .

In GR, this can be alleviated by the inclusion of a suitable boundary layer, *i.e.* a finite surface energy-momentum tensor  $S_{ab}$ , as given by Eq. (50). Such a procedure may also prove helpful in the present context, although it shall not be pursued here: as it stands, the inability to suitably enforce the required matching shows that the gravitational collapse of a linearly minimally coupled homogeneous sphere is only possible if the Lagrangean density of a perfect fluid is given by  $\mathcal{L} = p$ , not  $\mathcal{L} = -\rho$ .

## IX. DISCUSSION AND OUTLOOK

In this work we have described the dynamics of gravitational collapse of a dust sphere under the influence of a linear non-minimal coupling, thus extending the familiar Oppenheimer-Snyder collapse. We have examined the different effects that arise due to the choice of Lagrangean density of matter, namely the use of  $\mathcal{L} = -\rho$  or  $\mathcal{L} = p$ . We main results are threefold:

- In the  $\mathcal{L} = -\rho$  scenario, the dynamics of gravitational collapse deviates from GR, due to the more

evolved dynamics. However, the usual dependence of the density on the scale factor  $\rho \sim a^{-3}$  remains, and point-like singularity with infinite density are attained.

The  $\mathcal{L} = p$  case is much more interesting: although the dynamics are qualitatively the same as in GR (with all relevant quantities shifted but the same governing equations), the energy-momentum tensor is not conserved. This leads to a modified dependence for the density and, as a result, a geometric point-like singularity (*i.e.* where the scalar curvature diverges) is attained with a finite density! Depending on the value of  $\epsilon$ , this can fall within the classical domain,  $\rho \rightarrow \rho_f \ll M_P^4$ , thus avoiding the need for a description of the quantum regime.

- Analogously to the well-known Gibbons-York-Hawking boundary terms, we have found that an additional contribution to the action functional on the hypersurface signaling the surface of the spherical body must be considered. Its Lagrangean density is of the form  $\mathcal{L}_{\partial V} = (1 + \epsilon\mathcal{L}/\kappa^4) K$ , with possible, undetermined, additional terms depending on the induced metric and the Lagrangean density of matter.

By varying these boundary terms with respect to the former, we showed that the extrinsic curvature is in general discontinuous across the boundary of the spherical body.

- The interior description of the gravitational collapse in the  $\mathcal{L} = p$  case is suitably embedded into the surrounding Schwarzschild spacetime via the continuity of the induced metric and the extrinsic curvature. This leads to a shift of the mass  $M$  of the spherical body (given by the gravitational potential away from it) with respect to the gravitational mass  $M_0$ : this modification depends not only on the coupling strength  $\epsilon$ , but also on the value of

the initial density  $\rho_0$ : as a result, different event horizons arise after collapse for stars with the same initial mass, but distinct radius — thus breaking the no-hair theorem.

The scenario  $\mathcal{L} = -\rho$  is not so well-behaved: the matching of the inner and outer spacetimes turns out to be unfeasible, unless highly unnatural, apparently arbitrary extra terms are added to the boundary action. From a physical point of view, this can be related to the non-vanishing effective pressure that arises due to the non-minimal coupling — thus recalling the similar matching problem found in the gravitational collapse of a homogeneous sphere with pressure in GR.

Future work should clearly address the issue of considering the collapse of a non-homogeneous sphere, as well as one endowed with initial angular momentum and/or charge. The consideration of a thin shell could also shed further insight as to the issues found with the spacetime matching in the  $\mathcal{L} = -\rho$  scenario, as well as providing a test for the validity of Birkhoff's theorem.

A more evolved, non-linear form for the non-minimal coupling will undoubtedly also provide for a more evolved phenomenology. In particular, we speculate that such generalization could lead to a rather interesting scenario, where the spherical body does not collapse to a vanishing size, but instead asymptotically attains a final, finite radius.

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